# Math 1020 Week 6

## Trigonometric Equation

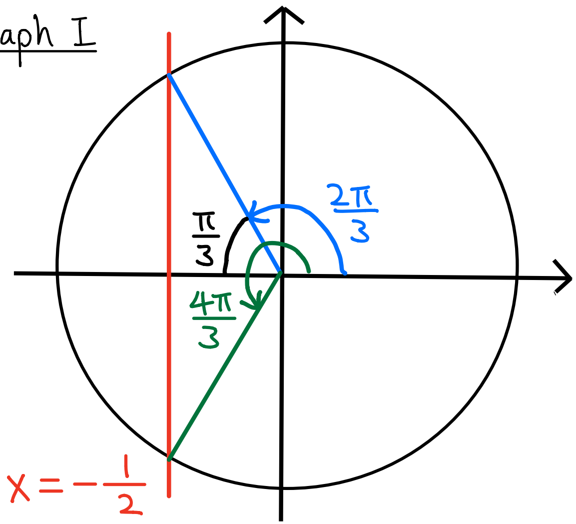
eg Solve $\cos\theta = -\frac{1}{2}$ for

① $0 \le \theta < 2\pi$ ② $\theta \in \mathbb{R}$

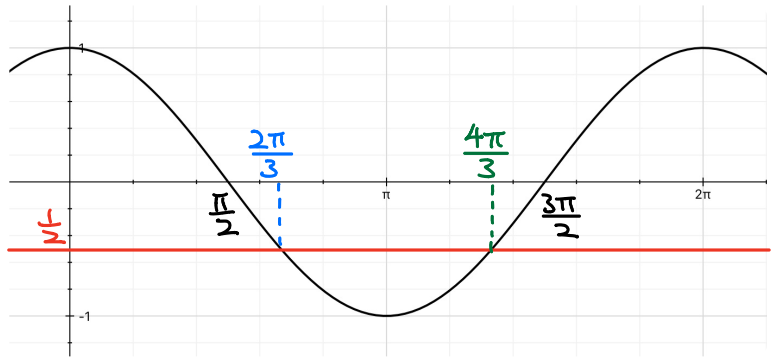
Ans ① $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

It can be deduced from either graph below

Graph I




Graph II




Rmk $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

② For $\theta \in \mathbb{R}$:

Note that $\cos\theta$ has period $2\pi$

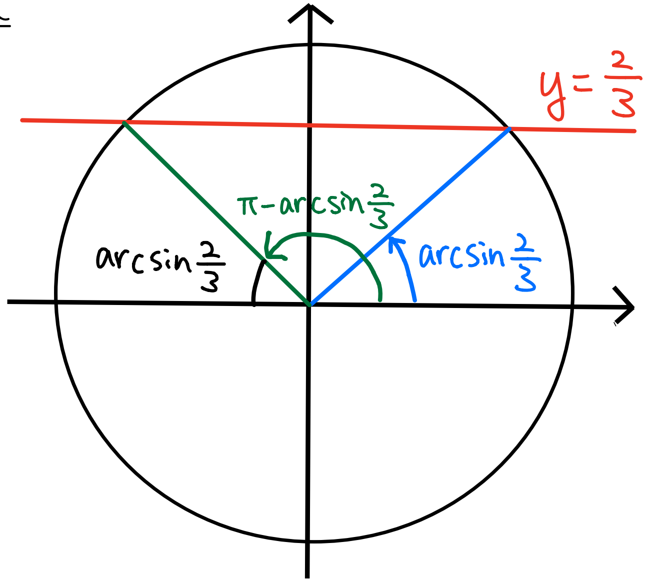
$\therefore \cos\theta = -\frac{1}{2}$

$\Longleftrightarrow \theta = \frac{2\pi}{3} + 2k\pi$ or $\frac{4\pi}{3} + 2k\pi$

where $k \in \mathbb{Z}$ (i.e. $k$ is an integer) eg Solve $\sin\theta = \frac{2}{3}$ for

① $\theta \in [0, 2\pi)$ ② $\theta \in \mathbb{R}$

Sol



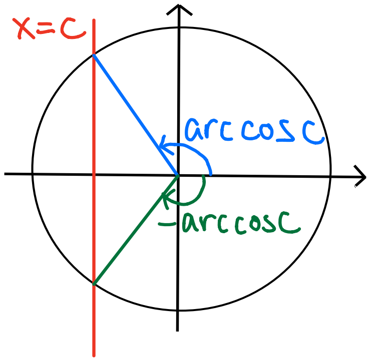

① $\theta = \arcsin\frac{2}{3}$ or $\pi - \arcsin\frac{2}{3}$

② $\theta = 2k\pi + \arcsin\frac{2}{3}$ or

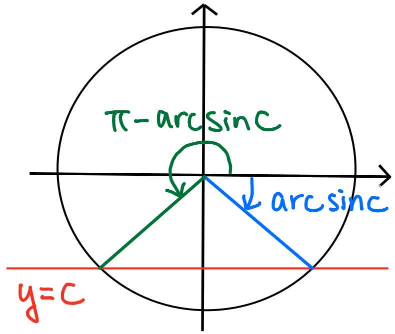
$(2k+1)\pi - \arcsin\frac{2}{3}$ where $k \in \mathbb{Z}$

In general, for $-1 \le c \le 1$ and $\theta \in \mathbb{R}$

$$\boxed{\cos\theta = c \Rightarrow \theta = 2k\pi \pm \arccos c, \quad k \in \mathbb{Z}}$$




$$\boxed{\begin{aligned} \sin\theta = c \Rightarrow \theta &= 2k\pi + \arcsin c \\ &\text{or } (2k+1)\pi - \arcsin c, \quad k \in \mathbb{Z} \end{aligned}}$$



eg Solve the following equations

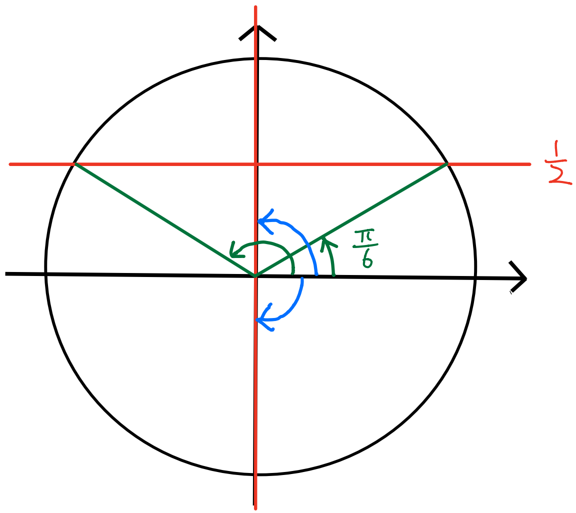
① $\sin 2x = \cos x, \quad x \in \mathbb{R}$

Sol $2\sin x \cos x = \cos x$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0 \quad$ or $\quad \sin x = \frac{1}{2}$

$x = 2k\pi \pm \frac{\pi}{2}$ or $2k\pi + \frac{\pi}{6}$ or $2k\pi + \frac{5\pi}{6}$




② $\cos^2 2\theta + 4\sin 2\theta = 4 \quad$ for $0 \le \theta \le 2\pi$

Sol $(1 - \sin^2 2\theta) + 4\sin 2\theta = 4$

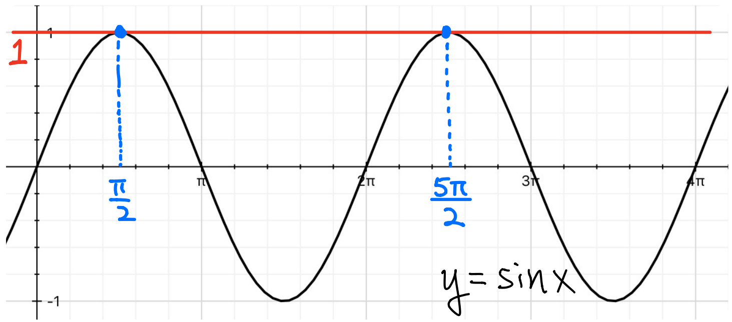
$-\sin^2 2\theta + 4\sin 2\theta - 3 = 0$

$\sin^2 2\theta - 4\sin 2\theta + 3 = 0$

$(\sin 2\theta - 1)(\sin 2\theta - 3) = 0$

$\sin 2\theta = 1 \quad$ or $\quad \sin 2\theta = 3$ (no solution)

Note $0 \le 2\theta \le 4\pi$




$\therefore 2\theta = \frac{\pi}{2}$ or $\frac{5\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

③ $\tan x + \tan 2x = 0$ for $x \in \mathbb{R}$

<u>Sol</u>

$$\tan x + \frac{2\tan x}{1-\tan^2 x} = 0$$

$$\tan x(1-\tan^2 x) + 2\tan x = 0$$

$$\tan x(3-\tan^2 x) = 0$$

$$\Rightarrow \tan x = 0 \quad \text{or} \quad \tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

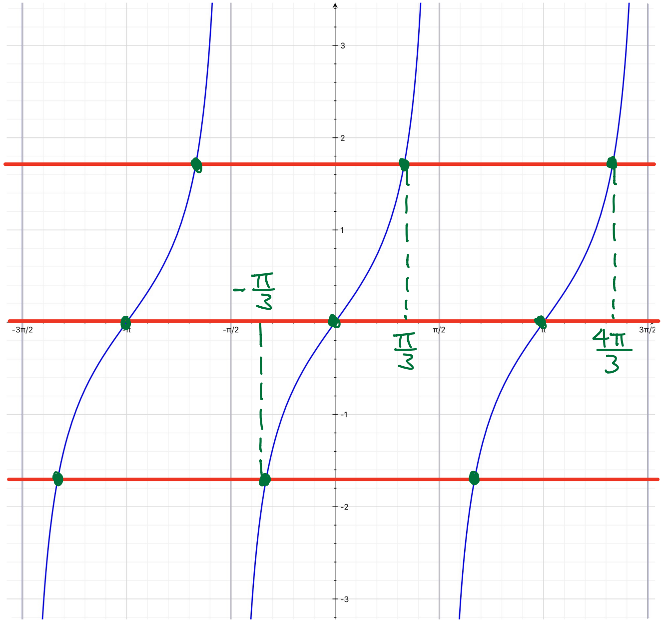
Note that $\tan x$ has period $\pi$

For $x \in \overbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}^{\text{length} = \pi}$, solutions are

$$x = 0 \quad \text{or} \quad \pm\frac{\pi}{3}$$

$\therefore$ General Solution :

$$x = k\pi,\ k\pi \pm \frac{\pi}{3} \quad \text{where } k \in \mathbb{Z}$$




In general, for $c \in \mathbb{R}$ and $\theta \in \mathbb{R}$

$$\boxed{\tan\theta = c \Rightarrow \theta = k\pi + \arctan c,\ k \in \mathbb{Z}}$$

⑥ $\sin x \sin 2x = \cos 3x \cos 4x \quad$ for $\quad 0 \le x \le \frac{\pi}{2}$

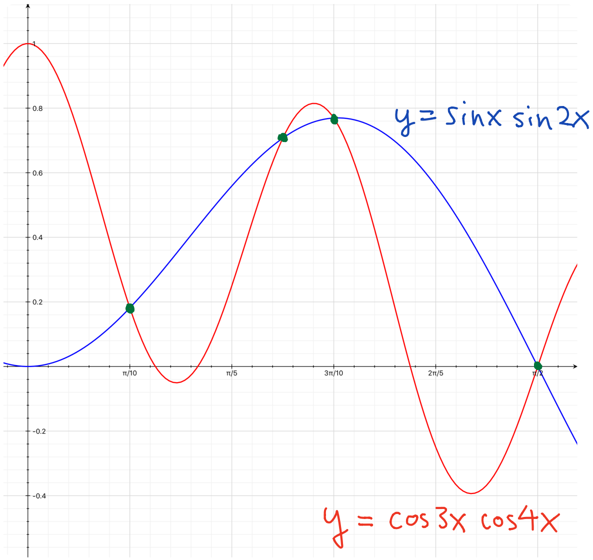
<u>Sol</u> $-\frac{1}{2}\left[\cos(x+2x) - \cos(x-2x)\right] = \frac{1}{2}\left[\cos(3x+4x) + \cos(3x-4x)\right]$ (product-to-sum formula)

$$-\cos 3x = \cos 7x$$

$$\cos 7x + \cos 3x = 0$$

$$2\cos\frac{7x+3x}{2}\cos\frac{7x-3x}{2} = 0$$ (Sum-to-product formula)

$$\cos 5x \cos 2x = 0$$




$\cos 5x = 0$ or $\cos 2x = 0$

$0 \le 5x \le \frac{5\pi}{2}$ $\qquad$ $0 \le 2x \le \pi$

$\Rightarrow 5x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ $\qquad$ $\Rightarrow 2x = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}$ $\qquad$ $\Rightarrow x = \frac{\pi}{4}$

$\therefore x = \frac{\pi}{10}, \frac{\pi}{4}, \frac{3\pi}{10}, \frac{\pi}{2}$

# Some more exercises in trigonometry

eg Let $f(x) = \sin x + \sqrt{3}\cos x$

i. Express $f(x) = A\sin(x+c)$ with $A \geq 0$

ii Find max/min value of $f(x)$ and the corresponding $x$

iii Solve $f(x) = 1$

Sol (i) Note

$$A\sin(x+c) = A(\sin x \cos c + \cos x \sin c)$$

$$= A\cos c \sin x + A \sin c \cos x$$

$$f(x) = 1 \sin x + \sqrt{3}\cos x$$

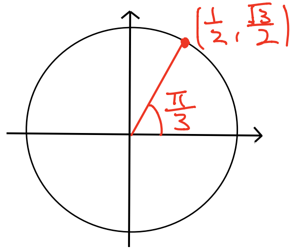
Want $\begin{cases} A\cos c = 1 \\ A\sin c = \sqrt{3} \end{cases}$ (*)

$\therefore (A\cos c)^2 + (A\sin c)^2 = 1^2 + \sqrt{3}^2$

$\Rightarrow A^2 = 4,\ A = 2 \quad (\because A \geq 0)$

$\therefore$ By (*), $\cos c = \frac{1}{2}$

$\sin c = \frac{\sqrt{3}}{2}$




We can take $c = \frac{\pi}{3}$

$\therefore f(x) = 2\sin(x + \frac{\pi}{3})$

(ii) From (i),

Max value $= 2$, occurs when

$$\sin(x+\tfrac{\pi}{3}) = 1 \Leftrightarrow x + \frac{\pi}{3} = 2k\pi + \frac{\pi}{2},\ k \in \mathbb{Z}$$

$$\Leftrightarrow x = 2k\pi + \frac{\pi}{6}$$

Min value $= -2$, occurs when

$$\sin(x+\tfrac{\pi}{3}) = -1 \Leftrightarrow x + \frac{\pi}{3} = 2k\pi - \frac{\pi}{2} \quad k \in \mathbb{Z}$$

$$\Leftrightarrow x = 2k\pi - \frac{5\pi}{6}$$

(iii) $f(x) = 1$

$$2\sin\left(x+\frac{\pi}{3}\right) = 1$$

$$\sin\left(x+\frac{\pi}{3}\right) = \frac{1}{2}$$

$$x+\frac{\pi}{3} = 2k\pi+\frac{\pi}{6} \text{ or } 2k\pi+\frac{5\pi}{6}$$

$$x = 2k\pi-\frac{\pi}{6} \text{ or } 2k\pi+\frac{\pi}{2}, \ k\in\mathbb{Z}$$

Ex Express

$$g(x) = \cos x - \sin x$$

$$h(x) = 3\cos x + 4\sin x$$

in the form $A\cos(x+c)$, $A \geq 0$

Ans: $g(x) = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

$$h(x) = 5\cos\left(x-\arctan\frac{4}{3}\right)$$

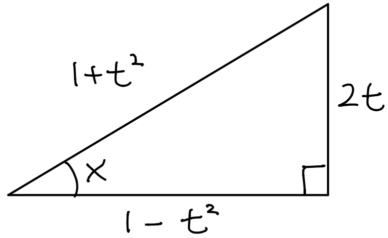
## t-substitution

Let $t = \tan\frac{x}{2}$. Then

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$




Picture when $0 < x < \frac{\pi}{2}$

Prove them!

eg

Express $\dfrac{\sin x+\cos x}{1+\sin x}$ as a rational function of $t$

Sol
$$\frac{\sin x+\cos x}{1+\sin x} = \frac{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}$$

$$= \frac{2t+1-t^2}{1+t^2+2t}$$

$$= \frac{-t^2+2t+1}{(1+t)^2}$$

# Trigonometric Substitution (for integration)

eg Express the followings in terms of $\theta$ with the given substitution and simplify

Assume $0 \le \theta \le \frac{\pi}{2}$

① $x\sqrt{25-x^2}$ , $x = 5\sin\theta$

$$= 5\sin\theta\sqrt{25-(5\sin\theta)^2}$$

$$= 5\sin\theta\sqrt{25(1-\sin^2\theta)}$$

$$= 5\sin\theta\cdot\sqrt{25\cos^2\theta}$$

$\because 0 \le \theta \le \frac{\pi}{2}$

$\cos\theta \ge 0$

$$= 5\sin\theta\cdot 5\cos\theta$$

$$= 25\sin\theta\cos\theta \text{ or } \frac{25}{2}\sin 2\theta$$

② $\dfrac{x}{\sqrt{4x^2+8x+5}}$ , $x+1 = \dfrac{\tan\theta}{2}$

$$= \frac{x}{\sqrt{4(x^2+2x+1)+1}}$$

$$= \frac{x}{\sqrt{4(x+1)^2+1}}$$

$$= \frac{\frac{\tan\theta}{2}-1}{\sqrt{\tan^2\theta+1}}$$

$$= \frac{\tan\theta-2}{2\sqrt{\sec^2\theta}}$$

$0 \le \theta \le \frac{\pi}{2}$

$\sec\theta \ge 0$

$$= \frac{\tan\theta-2}{2\sec\theta}$$

$$= \frac{1}{2}\cos\theta(\tan\theta-2)$$

$$= \frac{1}{2}(\sin\theta-\cos\theta)$$